裂项相消练习题

1 设$\{a_n\}$是等差数列，$\{b_n\}$是等比数列，已知$a_1=4$，$b_1=6$，$b_2=2a_2-2$，$b_3=2a_3+4$.

(1) 求$\{a_n\}$和$\{b_n\}$的通项公式 .

(2) 设数列$\{c_n\}$满足$c_1=1$，$c_n=\begin{cases}1,2^k<n<2^{k+1}\\ b_k,n=2^k\end{cases}$，其中$k\in\mathbf{N}^*$.

① 求数列$\{a_{2^n}(c_{2^n}-1)\}$的通项公式 .

② 求$\sum\limits_{i-1}^{2^n}a_ic_i\,(n\in\mathbf{N}^*)$.

答案 (1) $\{a_n\}$的通项公式为$a_n=3n+1$，$\{b_n\}$的通项公式为$b_n=3\cdot 2^n$.

(2) ① $a_{2^n}(c_{2^n}-1)=9\times 4^n-1$.

② $\sum\limits_{i=1}^{2^n}a_ic_i=27\times 2^{2n-1}+5\times 2^{n-1}-n-12\,(n\in\mathbf{N}^*)$.

解析 (1) 依题意得$\begin{cases}6q=6+2d\\ 6q^2=12+4d\end{cases}$，

解得$\begin{cases}d=3\\ q=2\end{cases}$，

故$\{a_n\}$的通项公式为$a_n=3n+1$，$\{b_n\}$的通项公式为$b_n=3\cdot 2^n$.

(2) ① $a_{2^n}(c_{2^n}-1)=a_{2^n}(b_n-1)=(3\times 2^n+1)(3\times 2^n-1)=9\times 4^n-1$，

所以，数列$\{a_{2^n}(c_{2^n}-1)\}$的通项公式为$a_{2^n}(c_{2^n}-1)=9\times 4^n-1$.

② $\sum\limits_{i=1}^{2^n}a_ic_i=\sum\limits_{i-1}^{2^n}[a_i+a_i(c_i-1)]$，

$=\sum\limits_{i=1}^{2^n}a_i+\sum\limits_{i-1}^{n}a_{2^i}(c_{2^i}-1)$，

$=\left(2^n\times 4+\dfrac{2^n(2^n-1)}{2}\times 3\right)+\sum\limits_{i-1}^{n}(9\times 4^i-1)$，

$=(3\times 2^{2n-1}+5\times 2^{n-2})+9\times\dfrac{4(1-4^n)}{1-4}-n$，

$=27\times 2^{2n-1}+5\times 2^{n-1}-n-12\,(n\in\mathbf{N}^*)$.

故$\sum\limits_{i=1}^{2^n}a_ic_i=27\times 2^{2n-1}+5\times 2^{n-1}-n-12\,(n\in\mathbf{N}^*)$.

2 设$\{a_n\}$是等比数列，公比大于0，其前n项和为$S_n\,(n\in\mathbf{N}^*)$，$\{b_n\}$是等差数列，已知$a_1=1$，$a_3=a_2+2$，$a_4=b_3+b_5$，$a_5=b_4+2b_6$.

(1) 求$\{a_n\}$和$\{b_n\}$的通项公式．

(2) 设数列$\{S_n\}$的前n项和为$T_n(n\in\mathbf{N}^*)$．

① 求T_n．

② 证明$\sum\limits_{k=1}^{n}\dfrac{(T_k+b_{k+2})b_k}{(k+1)(k+2)}=\dfrac{2^{n+2}}{n+2}-2(n\in\mathbf{N}^*)$．

答案 (1) $a_n=2^{n-1}$，$b_n=n$．

(2) ① $T_n=2^{n+1}-n-2$．

② 证明见解析．

解析 (1) 设等比数列$\{a_n\}$的公比q，

由$a_1=1$，$a_3=a_2+2$，可得$q^2-q-2=0$，

$\because q>0$，可得$q=2$，故$a_n=2^{n-1}$．

设等差数列$\{b_n\}$的公差为d，由$a_4=b_3+b_5$，

可得$b_1+3d=4$，由$a_5=b_4+2b_6$，

可得$3b_1+13d=16$，从而$b_1=1$，$d=1$，$\therefore b_n=2$．

$\therefore\{a_n\}$的通项公式为$a_n=2^{n-1}$，$\{b_n\}$的通项公式为$b_n=n$．

(2) ① 由(1)得，$S_n=\dfrac{1-2^n}{1-2}=2^n-1$，

$\therefore T_n=\sum\limits_{k=1}^{n}(2^k-1)=\sum\limits_{k=1}^{n}2^k-n=\dfrac{2\times(1-2^n)}{1-2}-n=2^{n+1}-n-2$．

② $\dfrac{(T_k+b_{k+2})b_k}{(k+1)(k+2)}=\dfrac{(2^{k+1}-k-2+k+2)k}{(k+1)(k+2)}$

$=\dfrac{k\cdot2^{k+1}}{(k+1)(k+2)}=\dfrac{2^{k+2}}{k+2}-\dfrac{2^{k+1}}{k+1}$，

$\therefore$

$\sum\limits_{k=1}^{n}\dfrac{(T_k+b_{k+2})b_k}{(k+1)(k+2)}=\left(\dfrac{2^3}{3}-\dfrac{2^2}{2}\right)+\left(\dfrac{2^4}{4}-\dfrac{2^3}{3}\right)+\cdots+\left(\dfrac{2^{n+2}}{n+2}-\dfrac{2^{n+1}}{n+1}\right)=\dfrac{2^{n+2}}{n+2}$
-2

．

3 已知数列$\{a_n\}$的前n项和为S_n，且满足$S_n=2a_n-1(n\in\mathbf{N}^*)$．

(1) 求数列$\{a_n\}$的通项公式．

(2) 证明：$\sum\limits_{k=1}^{n}\dfrac{1}{a_k^2}<\dfrac{4}{3}$．

答案 (1) $a_n=2^{n-1}$，$n\in\mathbf{N}^*$．

(2) 证明见解析．

解析 (1) 由题，得

当$n=1$时，$a_1=S_1=2a_1-1$，得$a_1=1$；

当$n\geqslant 2$时，$a_n=S_n-S_{n-1}=2a_n-1-2a_{n-1}+1$，

整理，得$a_n=2a_{n-1}$，

$\therefore$数列$\{a_n\}$是以1为首项，2为公比的等比数列，

$\therefore a_n=1\cdot 2^{n-1}=2^{n-1}$，$n\in\mathbf{N}^*$．

(2) 由（1）知，$\dfrac{1}{a_n^2}=\dfrac{1}{\left(2^{n-1}\right)^2}=\dfrac{1}{4^{n-1}}=\left(\dfrac{1}{4}\right)^{n-1}$，

故$\displaystyle\sum_{k=1}^{n}\frac{1}{a_k^2}=\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2}$

$=1+\left(\dfrac{1}{4}\right)^1+\left(\dfrac{1}{4}\right)^2+\cdots+\left(\dfrac{1}{4}\right)^{n-1}$

$=\dfrac{1-\left(\dfrac{1}{4}\right)^n}{1-\dfrac{1}{4}}$

$=\dfrac{4}{3}-\dfrac{4}{3}\cdot\left(\dfrac{1}{4}\right)^n<\dfrac{4}{3}$，

故得证．

4 设$\{a_n\}$是各项都为整数的等差数列，其前n项和为S_n，$\{b_n\}$是等比数列，且$a_1=b_1=1$，$a_3+b_2=7$，$S_5b_2=50$，$n\in\mathbf{N}^*$．

(1) 求数列$\{a_n\}$，$\{b_n\}$的通项公式．

(2) 设$c_n=\log_2 b_1+\log_2 b_2+\log_2 b_3+\cdots+\log_2 b_n$，$T_n=a_{c_n+1}+a_{c_n+2}+a_{c_n+3}+\cdots+a_{c_n+n}$．

① 求T_n．

② 求证：$\displaystyle\sum_{i=2}^{n}\frac{1}{\sqrt{T_i-i}}<2$．

答案 (1) $a_n=2n-1$，$b_n=2^{n-1}$．

(2) ① $T_n=n^3$．

② 证明见解析．

解析 (1) 设$\{a_n\}$的公差为d，$\{b_n\}$的公比为q，

依题意，$S_5b_2=5\left(1+2d\right)q=50$，$a_3+b_2=\left(1+2d\right)+q=7$，

解得$d=2$，$q=2$或$d=\dfrac{1}{2}$，$q=5$，

由于$\{a_n\}$是各项都为整数的等差数列，所以$d=2$，$q=2$．

从而$a_n=2n-1$，$b_n=2^{n-1}$．

(2) ① $\because \log_2 b_n=n-1$，

$\therefore c_n=0+1+2+\cdots+n-1=\dfrac{n\left(0+n-1\right)}{2}=\dfrac{n^2-n}{2}$，

$\therefore a_{c_n+i}=2\left(\dfrac{n^2-n}{2}+i\right)-1=n^2-n-1+2i$，

$\therefore T_n=n^2-n-1+2+n^2-n-1+4+\cdots+n^2-n-1+2n$

$=n\left(n^2-n-1\right)+\left(2+4+\cdots+2n\right)$

$=n\left(n^2-n-1\right)+n\left(n+1\right)=n^3$．

② $\because \dfrac{1}{\sqrt{T_n-n}}=\dfrac{1}{\sqrt{\left(n-1\right)n\left(n+1\right)}}$

$=\left(\dfrac{1}{\sqrt{\left(n-1\right)n}}-\dfrac{1}{\sqrt{n\left(n+1\right)}}\right)\dfrac{1}{\sqrt{n+1}-\sqrt{n-1}}$

$=\dfrac{1}{\sqrt{n}}\left(\dfrac{1}{\sqrt{n-1}}-\dfrac{1}{\sqrt{n+1}}\right)\dfrac{\sqrt{n+1}+\sqrt{n-1}}{2}$，

而$\dfrac{\sqrt{n+1}+\sqrt{n-1}}{2}<\sqrt{\dfrac{n+1+n-1}{2}}=\sqrt{n}$，

$\therefore \dfrac{1}{\sqrt{T_n-n}}<\dfrac{1}{\sqrt{n-1}}-\dfrac{1}{\sqrt{n+1}}$，

$\therefore \sum\limits_{i=2}^{n}\dfrac{1}{\sqrt{T_i-i}}<\dfrac{1}{\sqrt{2-1}}-\dfrac{1}{\sqrt{2+1}}+\dfrac{1}{\sqrt{3-1}}-\dfrac{1}{\sqrt{3+1}}+\dfrac{1}{\sqrt{4-1}}-\dfrac{1}{\sqrt{4+1}}$

$+\cdots+\dfrac{1}{\sqrt{n-2}}-\dfrac{1}{\sqrt{n}}+\dfrac{1}{\sqrt{n-1}}-\dfrac{1}{\sqrt{n+1}}$

$=1+\dfrac{\sqrt{2}}{2}-\dfrac{1}{\sqrt{n}}-\dfrac{1}{\sqrt{n+1}}<2$．

5 已知数列$\{a_n\}$的前n项和$S_n=-a_n-\left(\dfrac{1}{2}\right)^{n-1}+2(n\in\mathbf{N}^*)$，数列$\{b_n\}$满足$b_n=2^na_n$．

(1) 求证：数列$\{b_n\}$是等差数列，并求数列$\{a_n\}$的通项公式；

(2) 设$c_n=\dfrac{n(n+1)}{2^n(n-a_n)(n+1-a_{n+1})}$，数列$\{c_n\}$的前$n$项和为$T_n$，求满足$T_n<\dfrac{124}{63}(n\in\mathbf{N}^*)$的$n$的最大值．

答案 (1) $a_n=\dfrac{n}{2^n}$．

(2) n的最大值为4.

解析　(1) 在$S_n=-a_n-\left(\frac{1}{2}\right)^{n-1}+2$中，令$n=1$，

可得$S_1=-a_1-1+2$，即$a_1=\frac{1}{2}$.

当$n\geqslant 2$时，$S_{n-1}=-a_{n-1}-\left(\frac{1}{2}\right)^{n-2}+2$，

$\therefore a_n=S_n-S_{n-1}=-a_n+a_{n-1}+\left(\frac{1}{2}\right)^{n-1}$，

$\therefore 2a_n=a_{n-1}+\left(\frac{1}{2}\right)^{n-1}$，即$2^n a_n=2^{n-1}a_{n-1}+1$.

$\because b_n=2^n a_n$，$\therefore b_n=b_{n-1}+1$，即当$n\geqslant 2$时，$b_n-b_{n-1}=1$.

又$b_1=2a_1=1$，

$\therefore$数列$\{b_n\}$是首项和公差均为1的等差数列 .

于是$b_n=1+(n-1)\cdot 1=n=2^n a_n$，$\therefore a_n=\frac{n}{2^n}$.

(2) $\because c_n=\dfrac{n(n+1)}{2^n\left(n-a_n\right)\left(n+1-a_{n+1}\right)}=\dfrac{n(n+1)}{2^n\left(n-\frac{n}{2^n}\right)\left(n+1-\frac{n+1}{2^{n+1}}\right)}$

$=\dfrac{2^{n+1}}{\left(2^n-1\right)\left(2^{n+1}-1\right)}=\dfrac{2}{2^n-1}-\dfrac{2}{2^{n+1}-1}$，

$\therefore T_n=\left(\dfrac{2}{2^1-1}-\dfrac{2}{2^2-1}\right)+\left(\dfrac{2}{2^2-1}-\dfrac{2}{2^3-1}\right)+\cdots$
$+\left(\dfrac{2}{2^{n-1}-1}-\dfrac{2}{2^n-1}\right)+\left(\dfrac{2}{2^n-1}-\dfrac{2}{2^{n+1}-1}\right)$
$=\dfrac{2}{2^1-1}-\dfrac{2}{2^{n+1}-1}=2-\dfrac{2}{2^{n+1}-1}$.

由$T_n<\dfrac{124}{63}$，得$2-\dfrac{2}{2^{n+1}-1}<\dfrac{124}{63}$，即$\dfrac{1}{2^{n+1}-1}>\dfrac{1}{63}$，

$\therefore 2^{n+1}<64$，即$n<5$，

$\therefore n$的最大值为4.